

Adjacent q -Cycles in Permutations

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Abstract

We introduce a new permutation statistic, namely, the number of cycles of length q consisting of consecutive integers, and consider the distribution of this statistic among the permutations of $\{1, 2, \dots, n\}$. We determine explicit formulas, recurrence relations, and ordinary and exponential generating functions. A generalization to more than one fixed length is also considered.

Key words: permutations, derangements, adjacent cycles.

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1 Introduction

Let S_n denote the set of all permutations of $[n] = \{1, 2, \dots, n\}$. We use the one-line notation for permutations but then write a permutation according to its standard (disjoint) cycle decomposition. For example,

$$\pi = 432157869 = (14)(23)(5)(678)(9)$$

is a permutation in S_9 decomposed into cycles of lengths 2,2,1,3,1. (We omit commas between the integers in a cycle unless it leads to ambiguity.) We standardize the cycle decomposition by writing, for each cycle, its smallest element first and then writing the cycles in order of their smallest element. Generalizing the notion of an adjacent transposition (a 2-cycle), we define an *adjacent q -cycle*, abbreviated *AqC*, to be a q -cycle in which the elements in the cycle form a consecutive set of integers:

$$(a, a+1, a+2, \dots, a+q-1).$$

The permutation π above has two A1C's (the fixed points (5) and (9)), one A2C (the adjacent transposition (23)), and one A3C ((678)). A permutation in S_n is *adjacent q -cycle free*, abbreviated *AqC-free*, provided that it does not have any AqC's. For example, $532146 = (154)(23)(6)$ is A3C-free. An A1C-free permutation is just a usual derangement.

We introduce a new permutation statistic for S_n by considering, for an integer q with $1 \leq q \leq n$, the *number of AqC's* contained in a permutation, and consider the distribution of this statistic on S_n . More precisely, let

$$a(n, k) = |\{\pi \in S_n : \pi \text{ has exactly } k \text{ AqC's}\}|.$$

Let $r = \left\lfloor \frac{n}{q} \right\rfloor$. If $k > r$, then $a(n, k) = 0$. We are interested in the sequence of numbers

$$a(n, 0), a(n, 1), a(n, 2), \dots, a(n, r), \tag{1}$$

where note that, for simplicity, the parameter q is not reflected in the notation. For example, if $n = 3$ and $q = 2$, then $r = 1$ and we have $a(3, 0) = 4$ and $a(3, 1) = 2$, with these numbers coming from, respectively, the permutations

$$(1)(2)(3), (13)(2), (123), (132) \text{ and } (1)(23), (12)3.$$

If $q = 1$, and so $r = n$, the sequence (1) gives the number of permutations with a specified number of fixed points, and these numbers are the *rencontres numbers* (see p. 59 in [4]):

$$a(n, 0), a(n, 1), a(n, 2), \dots, a(n, n) \quad (q = 1).$$

The sequence of *derangement numbers* d_0, d_1, d_2, \dots (see e.g. Sec. 6.3 in [2]) is the sequence

$$a(0, 0), a(1, 0), a(2, 0), \dots \quad (q = 1)$$

and, as a simple combinatorial argument shows,

$$a(n, k) = \binom{n}{k} d_{n-k} \quad (q = 1).$$

We now summarize the contents of this paper. In Section 2, we derive a formula for the numbers $a(n, k)$ and use this formula to obtain $a(n, k)$ from the numbers $a(n, k - 1)$; we also discuss the number of AqC-free permutations. In Section 3, we derive a homogeneous recurrence relation for $a(n, k)$. In Section 4, we determine the ordinary and exponential generating functions for the sequence (1). In Section 5, we obtain a formula for the number of permutations having a specified number of adjacent cycles of each of several lengths. Finally, in Section 6, we illustrate how the permanent function can be used to calculate some of our numbers.

2 A formula

Again we fix q throughout this section and suppress it in our notation.

Theorem 2.1 *We have*

$$a(n, k) = \sum_{j=k}^{\lfloor \frac{n}{q} \rfloor} (-1)^{k+j} \binom{j}{k} \frac{(n - (q-1)j)!}{j!}, \quad \left(k = 0, 1, \dots, \left\lfloor \frac{n}{q} \right\rfloor \right). \quad (2)$$

Proof. We again set $r = \lfloor n/q \rfloor$, and we first determine a system of $r+1$ linear equations that the numbers $a(n, 0), a(n, 1), \dots, a(n, r)$ satisfy. This system is the triangular system:

$$\begin{aligned} \binom{0}{0} a(n, 0) + \binom{1}{0} a(n, 1) + \binom{2}{0} a(n, 2) + \dots + \binom{r}{0} a(n, r) &= \frac{n!}{0!} \\ \binom{1}{1} a(n, 1) + \binom{2}{1} a(n, 2) + \dots + \binom{r}{1} a(n, r) &= \frac{(n - (q-1))!}{1!} \\ \binom{2}{2} a(n, 2) + \dots + \binom{r}{2} a(n, r) &= \frac{(n - 2(q-1))!}{2!} \\ \dots &= \dots \\ \binom{r}{r} a(n, r) &= \frac{(n - r(q-1))!}{r!} \end{aligned} \quad (3)$$

To verify that the equations (4) hold, let t be an integer with $0 \leq t \leq r$ and let \mathcal{M}_n^t be the multiset of permutations π in S_n obtained by (i) selecting t AQC's, thereby filling in tq of the positions in the one-line representation of π , and then (ii) filling in the remaining $n - tq$ positions in all possible ways. This indeed defines a multiset and not a set since, for instance, if $n = 6$, $q = 2$, and $t = 2$, the permutation 214365 occurs three times: choose in (i) any pair of the 2-cycles (12), (34), and (56) (say (12) and (56) giving 21 - -65), and then fill in the remaining two positions to get 214365.

We now determine the cardinality of the multiset \mathcal{M}_n^t by counting in two different ways, and then equate the two counts to yield (3). Suppose that t AQC's have been placed in the one-line representation of a permutation. Let y_1, y_2, \dots, y_{t+1} be the number of contiguous empty positions before, in-between, and after the occupied positions. For example, taking $n = 13$, $q = 2$, and $t = 3$, if the selected $t = 3$ A2C's are (45), (67), and (ab) , where $a = 12$ and $b = 13$, then we have

$$- - - 5476 - - - ba,$$

and therefore, $y_1 = 3, y_2 = 0, y_3 = 4$ and $y_4 = 0$. In general, we have

$$y_1 + y_2 + \dots + y_{t+1} = n - tq$$

and this equation has

$$\binom{n + t - tq}{t}$$

nonnegative integral solutions (see e.g. Sec 2.5 in [2]). Thus

$$|\mathcal{M}_n^t| = \binom{n + t - tq}{t} (n - tq)! = \frac{(n - t(q-1))!}{t!} \quad (t = 0, 1, \dots, r).$$

For $i = t, t+1, \dots, r$, let $\mathcal{M}_n^t(i)$ be the multisubset of \mathcal{M}_n^t consisting of all those permutations π with exactly i AqC's. Each permutation in $\mathcal{M}_n^t(i)$ has t initially placed AqC's and $i-t$ additional AqC's obtained by filling in the remaining positions. Such a permutation occurs $\binom{i}{t}$ times, since any t of the i AqC's can be the initially placed AqC's. Thus

$$|\mathcal{M}_n^t(i)| = \binom{i}{t} a(n, i).$$

Since

$$|\mathcal{M}_n^t| = \sum_{i=t}^r |\mathcal{M}_n^t(i)|,$$

we have

$$\sum_{i=t}^r \binom{i}{t} a(n, i) = \frac{(n - (q-1)t)!}{t!} \quad (t = 0, 1, \dots, r),$$

the equations of the triangular system (4). The inverse of the coefficient matrix $A = \left[\binom{j}{i} : 0 \leq i, j \leq r \right]$ is $A^{-1} = \left[(-1)^{j+i} \binom{j}{i} : 0 \leq i, j \leq r \right]$, and this gives the formula (2). \square

Special cases of formula (2) are the classical formulas for the derangement and rencontres numbers.

Corollary 2.2 *Let $q = 1$. The rencontres numbers $a(n, k)$ satisfy*

$$a(n, k) = \binom{n}{k} d_{n-k} \quad (k = 0, 1, \dots, n).$$

Proof. Taking $q = 1$ in (2),

$$a(n, k) = n! \sum_{j=k}^n (-1)^{k+j} \frac{1}{k!(j-k)!} = \frac{n!}{k!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!}.$$

If $k = 0$, we get

$$d_n = a(n, 0) = n! \sum_{i=0}^n \frac{(-1)^i}{i!}.$$

Thus, for the rencontres numbers we get again

$$a(n, k) = \binom{n}{k} d_{n-k}.$$

\square

For each value of q , the numbers $a(n, k)$ determine a Pascal-like “triangle.” For instance, with $q = 5$, $a(n, k)$ counts the number of permutations in S_n with k A5C's giving

the following data:

n	$k =$	0	1	2
0		1	0	0
1		1	0	0
2		2	0	0
3		6	0	0
4		24	0	0
5		119	1	0
6		718	2	0
7		5034	6	0
8		40296	24	0
9		362760	120	0
10		3628081	718	1
11		39911763	5034	3
12		478961292	40296	12
13		6226657980	362760	60

In the following theorem, we establish a simple relationship between the entries of two consecutive columns of these triangles.

Theorem 2.3 *We have*

$$a(n+q-1, k) = \begin{cases} \frac{1}{k} \left(a(n, k-1) + (-1)^{k+\frac{n}{q}} \binom{\frac{n}{q}}{k-1} \right), & \text{if } q|n \\ \frac{1}{k} a(n, k-1), & \text{otherwise.} \end{cases} \quad (4)$$

Proof. We use the formula (2) for $a(n, k)$ in Theorem 2.1 to get, after elementary manipulation and change of summation variable,

$$a(n+q-1, k) - \frac{1}{k} a(n, k-1) = \sum_{i=k-1}^{\lfloor \frac{n-1}{q} \rfloor} \lambda(q, n, k, i) - \sum_{i=k-1}^{\lfloor \frac{n}{q} \rfloor} \lambda(q, n, k, i), \quad (5)$$

where

$$\lambda(q, n, k, i) = \frac{(-1)^{i+k+1} (n - (q-1)i)!}{i!k} \binom{i}{k-1}.$$

If q does not divide n , then the upper summation limits in (5) are equal, implying that $a(n+q-1) = \frac{1}{k} a(n, k-1)$. If q divides n , then $\lfloor \frac{n}{q} \rfloor = \lfloor \frac{n-1}{q} \rfloor + 1$, and we obtain

$$a(n+q-1, k) - \frac{1}{k} a(n, k-1) = \lambda(q, n, k, \lfloor \frac{n}{q} \rfloor) = (-1)^{k+\frac{n}{q}} \binom{\frac{n}{q}}{k-1},$$

completing the proof. \square

Let $b_n = a(n, 0)$ be the number of AqC -free permutations in S_n . If $q = 1$, then $b_n = d_n$, the n th derangement number. Thus the numbers b_n include the derangement numbers as a special case. The formula (2) gives the following formula for b_n .

Corollary 2.4

$$b_n = \sum_{j=0}^{\lfloor \frac{n}{q} \rfloor} (-1)^j \frac{(n - (q-1)j)!}{j!}. \quad (6)$$

If $q = 1$, then we get the following classical formula for the derangement numbers.

$$b_n = n! \sum_{j=0}^n \frac{(-1)^j}{j!} = d_n.$$

Setting $k = 1$ and replacing n by $n - q$ in (4), we get the following relation for the entries of column $k = 1$ of our triangle in terms of the number of AqC-free permutations.

Corollary 2.5

$$a(n-1, 1) = \begin{cases} b_{n-q} + (-1)^{\frac{n}{q}} & \text{if } q|n \\ b_{n-q} & \text{otherwise.} \end{cases} \quad (7)$$

It is well known (see e.g. [1], p. 107) that $\lim_{n \rightarrow \infty} \frac{d_n}{n!} = \frac{1}{e}$. We determine the corresponding limit for $q \geq 2$ in the next theorem.

Theorem 2.6 *Let $q \geq 2$ be given. Then $\lim_{n \rightarrow \infty} \frac{b_n}{n!} = 1$.*

Proof. The total number of AqC's that occur in the permutations in S_n equals $(n+1-q)!$. This is because there are $n+1-q$ adjacent q -cycles that can be formed from $1, 2, \dots, n$ and each such cycle can be extended to $(n-q)!$ permutations in S_n . Let c_n be the number of permutations in S_n with at least one AqC. Then $c_n \leq (n+1-q)!$, and so

$$n! - (n+1-q)! \leq n! - c_n = b_n \leq n!.$$

Hence

$$1 - \frac{(n+1-q)!}{n!} \leq \frac{b_n}{n!} \leq 1.$$

Since $q \geq 2$, the result follows. \square

3 Recurrence Relations

In this section we determine a recurrence relation for the numbers $a(n, k)$. We first obtain a recurrence relation for the number $b_n = a(n, 0)$ of AqC-free permutations of $\{1, 2, \dots, n\}$.

There is a well known recurrence relation satisfied by the derangement numbers d_n , namely, $d_n - nd_{n-1} = (-1)^n$. We now show that this is a special case of a recurrence relation satisfied by b_n in general.

Theorem 3.1 *Let $q \geq 1$ be given. Then*

$$b_n - nb_{n-1} = \begin{cases} (q-1)b_{n-q} + q(-1)^{\frac{n}{q}}, & \text{if } q|n \\ (q-1)b_{n-q}, & \text{otherwise} \end{cases} \quad (n \geq 1). \quad (8)$$

Proof. The proof follows by substitution of the formula for the numbers b_n given in (6). We verify that the recurrence relation (8) holds when $q|n$ and leave the verification to the reader if $q \nmid n$. Assume that $q|n$, and let $r = n/q$. Then by (6), we have

$$b_n = \sum_{j=0}^r (-1)^j \frac{(n - (q-1)j)!}{j!} \text{ and } b_{n-1} = \sum_{j=0}^{r-1} (-1)^j \frac{(n-1 - (q-1)j)!}{j!}.$$

Hence

$$\begin{aligned} b_n - nb_{n-1} &= (-1)^r \frac{(n - (q-1)r)!}{r!} + \sum_{j=0}^{r-1} (-1)^j \frac{(n-1 - (q-1)j)!(n - (q-1)j - n)}{j!} \\ &= (-1)^r - (q-1) \sum_{j=1}^{r-1} (-1)^j \frac{(n-1 - (q-1)j)!}{(j-1)!} \\ &= (-1)^r + (q-1) \sum_{i=0}^{r-2} (-1)^i \frac{(n-q - (q-1)i)!}{i!}. \end{aligned} \quad (9)$$

From (6) we also get

$$b_{n-q} = \sum_{j=0}^{r-1} (-1)^j \frac{(n-q - (q-1)j)!}{j!} = (-1)^{r-1} + \sum_{j=0}^{r-2} (-1)^j \frac{(n-q - (q-1)j)!}{j!}. \quad (10)$$

Combining (9) and (10), we get

$$b_n - nb_{n-1} = (-1)^r + (q-1)(b_{n-q} - (-1)^{r-1}) = (q-1)b_{n-q} + q(-1)^r.$$

□

We now derive a recurrence relation for the numbers $a(n, k)$.

Theorem 3.2 *For $k \geq 1$,*

$$a(n+1, k) = a(n-q+1, k-1) + (n-qk+1)a(n, k) - a(n-q+1, k) + q(k+1)a(n, k+1). \quad (11)$$

Proof. We denote a permutation in S_n with k AqC's by $\pi_{n,k}$. The permutations in S_{n+1} can be gotten from the permutations in S_n by inserting $n+1$ in any of the $n+1$ places before, between, and after the n entries. If one uses the cycle decomposition of a permutation, the permutation of S_{n+1} can be gotten from the permutations in S_n by

inserting $n + 1$ in any of the places after the entries of each cycle, or by creating a new cycle of length 1 (a fixed point). A permutation $\pi_{n+1,k}$ can be gotten in this way only from permutations $\pi_{n,k+1}$ (by killing one of the AqC 's), $\pi_{n,k}$ (by not killing its AqC 's and not creating any new AqC), and $\pi_{n,k-1}$ (by not killing its AqC 's and creating one new AqC). We now compute the number of $\pi_{n+1,k}$'s obtained in each of these three ways.

Case $\pi_{n+1,k}$'s from $\pi_{n,k+1}$'s: There are $k + 1$ q -cycles, each of which can be killed (by making it into a $(q + 1)$ -cycle) in q ways. In this way we get

$$q(k + 1)a(n, k + 1) \quad (12)$$

permutations with k AqC 's.

Case $\pi_{n+1,k}$'s from $\pi_{n,k}$'s: Here we have to distinguish two possibilities according to whether or not n belongs to an $A(q - 1)C$. Assume first that n belongs to an $A(q - 1)C$, so that $q \geq 2$ and the $A(q - 1)C$ containing n is $(n - q + 2, n - q + 3, \dots, n)$. Then $n + 1$ can be inserted as a cycle of length 1, or after each entry not in the k AqC 's except after the last entry of $(n - q + 2, n - q + 3, \dots, n)$. Thus, for each such $\pi_{n,k}$ we get $(1 + n - qk - 1)$ permutations with k AqC 's, giving a total of

$$(n - qk)a(n - q + 1, k)$$

permutations with k AqC 's.

If n does not belong to an $A(q - 1)C$, then the same reasoning, but without the exception, gives that there are $1 + (n - qk)$ positions in which to insert $n + 1$ to get a permutation with k AqC 's. In this case we get

$$(1 + n - qk)(a(n, k) - a(n - q + 1, k))$$

permutations with k AqC 's.

Adding these two numbers we get in this case

$$(1 + n - qk)(a(n, k) - a(n - q + 1, k)) \quad (13)$$

permutations with k AqC 's.

Case $\pi_{n+1,k}$'s from $\pi_{n,k-1}$'s: Now we have to create a new AqC by inserting $n + 1$ in a $\pi_{n,k-1}$, and this means that $(n - q + 2, n - q + 3, \dots, n)$ must be a $(q - 1)$ -cycle in $\pi_{n,k-1}$ and we have to insert $n + 1$ immediately after n . Thus in this case we get

$$a(n - k + 1, k - 1) \quad (14)$$

permutations with k AqC 's. Adding (12), (13), and (14), we obtain the recurrence (8).

□

4 Generating functions for A_qC -free permutations

The (ordinary) generating function $g(z)$ for derangements (A_1C -free permutations) is known (see p. 199, Exercise 4 in [3]) to satisfy the differential equation

$$z^2(1+z)g'(z) - (1-z^2)g(z) + 1 = 0, \quad (g(0) = 1).$$

Now let $q \geq 1$ and let

$$g(z) = \sum_{n=0}^{\infty} b_n z^n$$

be the (ordinary) generating function for the number b_n of A_qC -free permutations in S_n . Then $g(z)$ satisfies a first-order linear differential equation.

Theorem 4.1 *For every $q \geq 1$, $g(z)$ satisfies the differential equation*

$$z^2(1+z^q)g'(z) - (1+z^q)(1-z - (q-1)z^q)g(z) + 1 - (q-1)z^q = 0, \quad (g(0) = 1). \quad (15)$$

Proof. Start with the recurrence relation (8) satisfied by the b_n , multiply by z^n , and then sum over all $n \geq q$. \square

Now let

$$G(z) = \sum_{n=1}^{\infty} b_n \frac{z^n}{n!}$$

be the exponential generating function for the numbers b_n given in (6).

Theorem 4.2 *For every $q \geq 1$, $G(z)$ satisfies the q th-order differential equation*

$$(1-z)G^{(q)}(z) - qG^{(q-1)}(z) - (q-1)G(z) = qw^{(q)}(z), \quad (G^{(i)}(0) = i! \ (i = 0, 1, \dots, q-1)) \quad (16)$$

where

$$w(z) = \sum_{i=1}^{\infty} \frac{(-1)^i z^{qi}}{(qi)!}.$$

Proof. We make use of (8). We have

$$\frac{b_{qn}}{(qn)!} = \frac{b_{qn-1}}{(qn-1)!} + \frac{(q-1)}{qn(qn-1) \cdots (qn-q+1)} \frac{b_{qn-q}}{(qn-q)!} + \frac{q(-1)^n}{(qn)!}. \quad (17)$$

We also have for $j = 1, 2, \dots, q-1$,

$$\frac{b_{qn+j}}{(qn+j)!} = \frac{b_{qn+j-1}}{(qn+j-1)!} + \frac{(q-1)}{(qn+j)(qn+j-1) \cdots (qn-q+j+1)} \frac{b_{qn-q+j}}{(qn-q+j)!} \quad (18)$$

Multiplying (17) by z^{qn} , and multiplying, for $j = 1, 2, \dots, q-1$, the j th equation of (18) by z^{qn+j} , and then summing for $n \geq 0$, we get

$$G(z) - \sum_{i=0}^{q-1} \frac{b_i}{i!} z^i = z \left(G(z) - \sum_{i=0}^{q-2} \frac{b_i}{i!} z^i \right) + (q-1)z^q K(z) + q \sum_{i=1}^{\infty} \frac{(-1)^i z^{qi}}{(qi)!}, \quad (19)$$

where

$$K(z) = \sum_{i=0}^{\infty} \frac{b_i}{(i+1)(i+2) \cdots (i+q)} \frac{z^i}{i!}.$$

Differentiating (19) q times, and using the fact that the q th derivative of $z^q K(z)$ equals $G(z)$, we obtain (16). \square

The function $w(z)$ in Theorem 4.2 can be expressed as a hypergeometric function, namely

$$w(z) = -\frac{z^q}{q!} F \left(1; \frac{q+1}{q}, \frac{q+2}{q}, \dots, \frac{2q}{q}; -\frac{z^q}{q^q} \right).$$

If $q = 1$, we obtain the known (see e.g. p. 106 in [1]) exponential generating function for the derangements, namely $G(z) = \frac{e^{-z}}{1-z}$.

5 Adjacent cycles of more than one length

Up to now we have considered permutations focusing on adjacent cycles of one given length q . There is a natural generalization of some of the results in the previous sections obtained by replacing q with a finite set Q of $m \geq 1$ lengths. Thus let

$$Q = \{q_1, q_2, \dots, q_m\} \text{ where } 1 \leq q_1 < q_2 < \cdots < q_m.$$

Let

$$a(n; k_1, k_2, \dots, k_m) = |\{\pi \in S_n : \pi \text{ has } k_1 \text{ } Aq_1\text{C's, } k_2 \text{ } Aq_2\text{C's, } \dots, k_m \text{ } Aq_m\text{C's}\}|.$$

Note that $a(n; k_1, k_2, \dots, k_m) = 0$ if $q_1 k_1 + q_2 k_2 + \cdots + q_m k_m > n$. As for the case of one length, we suppress the lengths $Q = \{q_1, q_2, \dots, q_m\}$ in our notation. The following theorem is the multi-length analogue of Theorem 2.1.

Theorem 5.1 *We have*

$$a(n; k_1, k_2, \dots, k_m) = \sum_{\{t_1, t_2, \dots, t_m \geq 0 : \sum_{j=1}^m q_j t_j \leq n\}} (-1)^{\sum_{j=1}^m (k_j + t_j)} \prod_{j=1}^m \binom{t_j}{k_j} \frac{\left(n - \sum_{j=1}^m (q_j - 1)t_j\right)!}{\prod_{j=1}^m t_j!}. \quad (20)$$

Proof. The proof is analogous to the proof of Theorem 2.1 and so we omit some of the details. Moreover, we illustrate the proof only in the case that $m = 3$ from which it is clear how to proceed in general. For $m = 3$ we write (20) in the form

$$a(n; i, j, k) = \sum_{\{s, t, u \geq 0: ps + qt + ru \leq n\}} (-1)^{(i+j+k)+(s+t+u)} \binom{s}{i} \binom{t}{j} \binom{u}{k} \frac{(n - (p-1)s - (q-1)t - (r-1)u)!}{s!t!u!}. \quad (21)$$

Corresponding to the triangular system (3) we obtain in a similar way the system of equations valid for all s, t, u with $ps + qt + ru \leq n$:

$$\sum_{\{\alpha, \beta, \gamma \geq 0: p\alpha + q\beta + r\gamma \leq n\}} \binom{\alpha}{s} \binom{\beta}{t} \binom{\gamma}{u} a(n; \alpha, \beta, \gamma) = \frac{(n - (p-1)s - (q-1)t - (r-1)u)!}{s!t!u!}. \quad (22)$$

To find the solution of this system, we require the following lemma.

Lemma 5.2 *Let n, p, q, r be nonnegative integers with $p, q, r \leq n$, and let α, β, γ be non-negative integers with $p\alpha + q\beta + r\gamma \leq n$. Then*

$$\sum_{\{s, t, u \geq 0: ps + qt + ru \leq n\}} (-1)^{s+t+u} \binom{s}{i} \binom{\alpha}{s} \binom{t}{j} \binom{\beta}{t} \binom{u}{k} \binom{\gamma}{u} = \begin{cases} (-1)^{i+j+k} & \text{if } (\alpha, \beta, \gamma) = (i, j, k) \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

Proof of Lemma 5.2. The product

$$\binom{s}{i} \binom{\alpha}{s} \binom{t}{j} \binom{\beta}{t} \binom{u}{k} \binom{\gamma}{u}$$

is nonzero if and only if $i \leq s \leq \alpha$, $j \leq t \leq \beta$, and $k \leq u \leq \gamma$. All these triples belong to the summation in (23) because $ps + qt + ru \leq p\alpha + q\beta + r\gamma \leq n$. Thus the left hand side of (23) can be rewritten as a product of three sums:

$$\left(\sum_{s=i}^{\alpha} (-1)^s \binom{s}{i} \binom{\alpha}{s} \right) \left(\sum_{t=j}^{\beta} (-1)^t \binom{t}{j} \binom{\beta}{t} \right) \left(\sum_{u=k}^{\gamma} (-1)^u \binom{u}{k} \binom{\gamma}{u} \right).$$

Considering the first sum in this product, we get

$$\sum_{s=i}^{\alpha} (-1)^s \binom{s}{i} \binom{\alpha}{s} = (-1)^i \binom{\alpha}{i} \sum_{s=i}^{\alpha} (-1)^{s-i} \binom{\alpha-i}{s-i} = \begin{cases} 0 & \text{if } \alpha \neq i \\ (-1)^i & \text{if } \alpha = i, \end{cases}$$

since the last summation leads to the full alternating sum of the binomial coefficients of the form $\binom{\alpha-i}{*}$. The other two sums in the product are similarly evaluated giving (23), and thus proving the lemma.

Returning to the proof of the theorem, we multiply both sides of (22) by

$$(-1)^{i+j+k+s+t+u} \binom{s}{i} \binom{t}{j} \binom{u}{k}$$

and then sum over all triples (s, t, u) with $s, t, u \geq 0$ and $ps + qt + ru \leq n$. The result on the right hand side is the same as the right side of (21), while on the left hand side we get

$$\sum_{\{s,t,u \geq 0: ps+qt+ru \leq n\}} (-1)^{(i+j+k)+(s+t+u)} \binom{s}{i} \binom{t}{j} \binom{u}{k} \sum_{\{\alpha, \beta, \gamma \geq 0, p\alpha+q\beta+r\gamma \leq n\}} \binom{\alpha}{s} \binom{\beta}{t} \binom{\gamma}{u} a(\alpha, \beta, \gamma),$$

and this equals

$$(-1)^{i+j+k} \sum_{\{\alpha, \beta, \gamma \geq 0, p\alpha+q\beta+r\gamma \leq n\}} a(\alpha, \beta, \gamma) \sum_{\{s,t,u \geq 0: ps+qt+ru \leq n\}} (-1)^{s+t+u} \binom{s}{i} \binom{t}{j} \binom{u}{k} \binom{\alpha}{s} \binom{\beta}{t} \binom{\gamma}{u},$$

or $a(i, j, k)$ in view of Lemma (5.2). \square

By taking $k_1 = k_2 = \dots = k_m = 0$ in (20), we obtain the number of permutations in S_n having no adjacent cycles of any of the lengths q_1, q_2, \dots, q_m , namely,

$$a(n; 0, 0, \dots, 0) = \sum_{\{t_1, t_2, \dots, t_m \geq 0: \sum_{j=1}^m q_j t_j \leq n\}} (-1)^{\sum_{j=1}^m t_j} \frac{(n - \sum_{j=1}^m (q_j - 1)t_j)!}{\prod_{j=1}^m t_j!}.$$

If we take $(q_1, q_2, \dots, q_k) = (1, 2, \dots, k)$, then we obtain

$$a(n; 0, 0, \dots, 0) = \sum_{\{t_1, t_2, \dots, t_m \geq 0: \sum_{j=1}^m j t_j \leq n\}} (-1)^{\sum_{j=1}^m t_j} \frac{(n - \sum_{j=1}^m (j - 1)t_j)!}{\prod_{j=1}^m t_j!}.$$

These numbers can be viewed as *restricted derangement numbers* as the permutations they count are, in particular, derangements.

6 Concluding remarks

It is well known that, for a fixed n , the rencontre numbers $r(n, k)$ can be calculated using permanents. Specifically, let J_n be the all 1s matrix of order n , and let I_n be the identity matrix of order n . Then

$$\text{per}(xI_n + (J_n - I_n)) = \sum_{k=0}^n r(n, k) x^{n-k}$$

is the generating function for the numbers $r(n, 0) = d_n, r(n, 1), r(n, 2), \dots, r(n, n) = 1$ (see [4], pp. 59, 184). By constructing appropriate matrices, we can also use permanents to obtain, for fixed n and q , the generating polynomial for the numbers $a(n, 0), a(n, 1), a(n, 2), \dots, a(n, \lfloor \frac{n}{q} \rfloor)$.

This procedure is more complicated and, as a result, we do not formalize this but only illustrate the technique by a few examples.

Example. Consider $n = 6$ and $q = 3$ and the matrix

$$\begin{bmatrix} 1 & x_1 & 1 & 1 & 1 & 1 \\ 1 & 1 & x_1x_2 & 1 & 1 & \\ x_1 & 1 & 1 & x_2x_3 & 1 & 1 \\ 1 & x_2 & 1 & 1 & x_3x_4 & 1 \\ 1 & 1 & x_3 & 1 & 1 & x_4 \\ 1 & 1 & 1 & x_4 & 1 & 1 \end{bmatrix},$$

obtained by introducing an indeterminate for each A3C and placing it in the associated positions of the matrix. Calculating the permanent ($6! = 720$ terms), we get a polynomial $p(x_1, x_2, x_3, x_4)$ in x_1, x_2, x_3, x_4 . In order to obtain the number of terms in this permanent which use all three x_j 's for each j , we first make the substitutions $x_1^3 = x$, $x_2^3 = x$, $x_3^3 = x$, $x_4^3 = x$ in $p(x_1, x_2, x_3, x_4)$, and then make the substitutions $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, and $x_4 = 1$. This yields the degree 3 polynomial

$$697 + 22x + x^2.$$

Here $a(6, 0) = 697$, $a(6, 1) = 22$, and $a(6, 2) = 1$. That $a(6, 2) = 1$ follows since the only permutation in S_6 with two A3C's is $(123)(456)$. We verify that $a(6, 1) = 22$ as follows: There are 5 permutations whose unique A3C is (123) , 6 permutations whose unique A3C is (234) , 6 permutations whose unique A3C is (345) , and 5 permutations whose unique A3C is (456) . The remaining $6! - 1 - 22 = 697$ permutations have no A3C's.

In case we consider adjacent cycles of several lengths, that is, $Q = \{q_1, q_2, \dots, q_m\}$, we can extend the above procedure to obtain a multivariable generating polynomial for the numbers $a(n; k_1, k_2, \dots, k_m)$. To illustrate this technique, let $n = 5$ and let $Q = \{1, 2, 3, 4, 5\}$ (all possible lengths of adjacent cycles of permutations in S_5). Consider the matrix

$$\begin{bmatrix} x & y_1z_1u_1v_1 & 1 & 1 & 1 \\ y_1 & x & y_2z_1z_2u_1u_2v_1 & 1 & 1 \\ z_1 & y_2 & x & y_3z_2z_3u_1u_2v_1 & 1 \\ u_1 & z_2 & y_3 & x & y_4z_3u_2v_1 \\ v_1 & u_2 & z_3 & y_4 & x \end{bmatrix}.$$

The x 's, y_j 's, z_j 's, u_j 's, v_1 correspond to A1C's, A2C's, A3C's, A4C's, and A5C's, respectively. Let $p(x, y_1, y_2, \dots, v_1)$ be the multivariate polynomial obtained by calculating the permanent of this matrix. Replacing each y_j^2 with y , each z_j^3 with z , each u_j^4 with u , and v_1^5 with v , and then in the resulting polynomial, replacing each remaining subscripted variable with 1, we obtain

$$f(x, y, z, u, v) = 34 + 34x + 6y + z + v + 17x^2 + 6xy + 2xu + 2yz + 6x^3 + 3xy^2 + 3x^2z + 4x^3y + x^5,$$

the 5-variate generating polynomial for S_5 where x, y, z, u, v mark the A1C's, A2C's, A3C's, A4C's, and A5C's, respectively. For example, $3xy^2$ counts the permutations $(1)(23)(45)$, $(12)(3)(45)$, and $(12)(34)(5)$, while z counts the permutation $(15)(234)$.

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